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The solution of the contact problem for a circular plate☆

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ABSTRACT

An approach, based on Rektorys' theorem on the minimum of a quadratic functional which, without any fundamental difficulties, can be used for diverse contact problems, is used to solve the problem of the contact interaction of a circular flexible plate with an elastic half-space.

Detailed reviews of methods for analysing circular plates on an elastic base are available in Refs. 1–3, for example.

Consider a flexible plate of radius a on an elastic half-space with Poisson's ratio ν_0 and modulus of elasticity E_0 under the action of a cross-shaped load of intensity q, distributed over two mutually perpendicular diameters. The integral equation for determining of the reactive normal pressures in the contact area in the traditional formulation[1,2] can be written in the form

$$W(r,\varphi) = \frac{1-\nu_0^2}{\pi E_0}\int_0^{2\pi}\int_0^{a} p(\rho,\theta)K(r,\rho,\varphi,\theta)\rho d\rho d\theta \tag{1}$$

where $p(\rho,\theta)$ is the unknown law of the distribution of the reactive pressures and $W(r,\varphi)$ are the settlements of the elastic base, which are equal to the deflections of the plate.

In the case of an elastic, homogeneous, isotropic half-space[3]

$$K(r,\rho,\varphi,\theta) = [r^2 - 2r\rho\cos(\varphi-\theta)+\rho^2]^{-1/2} \tag{2}$$

This function is represented in the form of integrals containing Bessel functions.[4]

In the case of a cross-shaped load, we use for the distribution of the reactive pressures in the contact area,

$$p(\rho,\theta) = p_0(\rho) + p_4(\rho)\cos 4\theta + p_8(\rho)\cos 8\theta + \dots \tag{3}$$

$$p_{4n}(\rho) = \left(1-\frac{\rho^2}{a^2}\right)^{-1/2}\sum_{m=2n}^{\infty} H_{2m}^{4n}P_{2m}^{4n}\left(\sqrt{1-\frac{\rho^2}{a^2}}\right),$$
$$n = 0, 1, 2, \dots \tag{4}$$

where $P_k^l(z)$ is a spherical function[4] and H_{2m}^{4n} are unknown coefficients.

We substitute expressions (2) and (3) into Eq. (1) and integrate with respect to θ and ρ using the values of known integrals (Ref. 4, formula 6.574(1), Ref. 5, formula (6), and Ref. 6, formula 7.3.1.40). We obtain

$$W(r,\varphi) = \frac{1}{\pi}(\Sigma_0+\Sigma_1+\Sigma_2+\dots)$$

$$\Sigma_n = \frac{1-\nu_0^2}{E_0}a\sum_{k=2n}^{\infty} H_{2k}^{4n}\frac{\Gamma(k+2n+1/2)\Gamma(k-2n+1/2)}{\Gamma(k+2n+1)\Gamma(k-2n+1)}P_{2k}^{4n}\left(\sqrt{1-\frac{r^2}{a^2}}\right)\cos 4n\varphi,$$
$$n = 0, 1, 2, \dots \tag{5}$$

Following the well-known approach,[7] we represent the settlements of the circular plate in the form of a series in the eigenfunctions of the differential operator for the flexural oscillations of a circular plate with free edges, that is,

$$W(r,\varphi) = W_0(r) + W_4(r)\cos 4\varphi + W_8(r)\cos 8\varphi + \dots$$

$$W_{4m}(r) = \sum_{i=1}^{\infty} W_{4m,i} = \sum_{i=1}^{\infty} A_{4m,i}\chi_{4m,i}, \quad m = 0, 1, 2, \dots$$

$$\chi_{4m,i} = J_{4m}\left(\lambda_{4m,i}\frac{r}{a}\right) + C_{4m,i}I_{4m}\left(\lambda_{4m,i}\frac{r}{a}\right), \quad \chi_{0,1} = \sqrt{2} \tag{6}$$

Here, $J_{4m}(z)$, $I_{4m}(z)$ are Bessel functions[4], and the coefficients $A_{4m,i}$, $C_{4m,i}$ depend on λ_{4m} and Poisson's ratio of the plate (see Ref. 7).

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We substitute expressions (6) into equality (5) and make use of the properties of Fourier series. We obtain a system of functional relations for determining the unknown coefficients

$$\sqrt{2}A_{0,1} + \sum_{i=2}^{\infty} A_{0,i}\chi_{0,i} = \Sigma_0, \quad \sum_{i=1}^{\infty} A_{4n,i}\chi_{4n,i} = \Sigma_n,$$
$$n = 1, 2, 3, \ldots \tag{7}$$

The first of these equations corresponds to the axisymmetric contact problem for a circular plate, and the coefficient $A_{0,1}$ corresponds to the translational displacement of a rigid circular punch.

Equations (7) are solved using the standard procedure of the method of orthogonal polynomials.[3] Both sides of the equations are multiplied by $P_{2m}^{4n}(\sqrt{1-r^2/a^2})rdr/\sqrt{1-r^2/a^2}$ and integrated in the limits from 0 to a ($n=0, 1, 2, \ldots$); ($m=2n, 2n+1.2n+2, \ldots$). The orthogonality properties of spherical functions and relation (5) are also used. As a result, we obtain the equations for the relation between $A_{4n,i}$ and H_{2k}^{4n}

$$H_0^0 = \frac{E_0}{\pi(1-\nu_0^2)a}\left[\sqrt{2}A_{0,1} + \sum_{i=2}^{\infty} A_{0,i}F_0^0(\lambda_{0,i})\right]$$
$$H_{2n}^0 = \frac{E_0}{(1-\nu_0^2)a}\sum_{i=1}^{\infty} A_{0,i}\frac{(4n+1)\Gamma^2(n+1/2)}{\Gamma^2(n+1)}F_{2n}^0(\lambda_{0,i})$$
$$H_{4n+2m}^{4n} = \frac{E_0}{(1-\nu_0^2)a}\sum_{i=1}^{\infty} A_{4n,i}\frac{(4m+8n+1)\Gamma(m+4n+1)\Gamma(m+1)}{\Gamma(m+2n+5/2)\Gamma(m+1/2)} F_{4n+2m}^{4n}(\lambda_{4n,i})$$
$$F_{4n+2k}^{4n}(\lambda_{4n,i}) = \frac{1}{a^2}\int_0^a \chi_{4n,i}\frac{P_{4n+2k}^{4n}(\sqrt{1-r^2/a^2})}{1-r^2/a^2}rdr;$$
$$n = 1, 2, \ldots \tag{8}$$

We now use Rektorys' approach.[8] As applied to the problem in question, the system of Rektorys' equations will have a form which is identical for each n (in view of the uniformity of Eq. (7) and the eigenfunctions, the case when $n=4$ is considered below):

$$\begin{Vmatrix} R_{11} & R_{12} & \ldots \\ R_{21} & R_{22} & \ldots \\ & \ldots & \end{Vmatrix}\begin{Vmatrix} A_{4,1} \\ A_{4,2} \\ \ldots \end{Vmatrix} = \begin{Vmatrix} \left(W_{4,1}, \frac{q(r,\varphi) - p_4(r)\cos 4\varphi}{D}\right) \\ \left(W_{4,2}, \frac{q(r,\varphi) - p_4(r)\cos 4\varphi}{D}\right) \\ \ldots \end{Vmatrix} \tag{9}$$

Here, $R_{i,j} = (\Delta^2 W_{4,i}, W_{4,j})$ and D is the cylindrical stiffness of the plate.

By virtue of the orthogonality of the functions adopted for the flexures of the plate, we have

$$R_{i,i} = a^{-2}\lambda_{4,i}^4 W_{4,i}; \quad R_{i,j} = 0, \quad i \neq j$$
$$\left(W_{4,i}\frac{p_4(r)\cos 4\varphi}{D}\right) = \int_0^{2\pi}\cos^2 4\varphi d\varphi \int_0^a \chi_{4,i}\sum_{m=0}^{\infty} H_{2m+4}^4 P_{2m+4}^4(\sqrt{1-r^2/a^2})rdr = \sum_{m=0}^{\infty} H_{2m+4}^4 F_{2m+4}^4(\lambda_{4,i}) \tag{10}$$

Then, taking account of the expression for H_{2m+4}^4 (the third formula of (8)), we obtain

$$\left(W_{4,i}\frac{p_4(r)\cos 4\varphi}{D}\right) = \frac{\pi E_0 a}{D(1-\nu_0^2)}\sum_{j=1}^{\infty} A_{4,j} \sum_{m=0}^{\infty}\frac{(4m+9)\Gamma(m+5)\Gamma(m+1)}{\Gamma(m+1/2)\Gamma(m+9/2)}F_{2m+4}^4(\lambda_{4,j})F_{2m+4}^4(\lambda_{4,i}) \tag{11}$$

In order to calculate $(W_{4,i}, q(r, \varphi)/D)$, we represent the cross-shaped load in the form of a load with an intensity $q/\alpha a$, uniformly distributed over the four sectors with an angle 2α, where the axes of symmetry of the neighbouring sectors make an angle of $\pi/2$. Taking the limit as $\alpha \to 0$, we obtain

$$\left(W_{4,i}\frac{q_4(r,\varphi)}{D}\right) = qa\frac{\lambda_{4,i}^4}{12\Gamma(5)}\left[F_2\left(3; 4, 5; -\frac{\lambda_{4,i}^2}{4}\right) + C_{4,i1}F_2\left(3; 4, 5; \frac{\lambda_{4,i}^2}{4}\right)\right] \tag{12}$$

Expressions (10)–(12) enables us to put together system (9), the solution of which gives the coefficients $A_{4,i}$.

As an example, consider a circular plate with a flexibility index $\beta = 1000$ on an elastic base. When

$\lambda_{0,1} = 0, \quad \lambda_{0,2} = 3.0005, \quad \lambda_{0,3} = 6.2003,$

$\lambda_{0,4} = 9.3675, \quad \lambda_{0,5} = 12.5227, \ldots$

$\lambda_{4,1} = 4.6728, \quad \lambda_{4,2} = 8.5757, \quad \lambda_{4,3} = 11.9344,$

$\lambda_{4,4} = 15.1997, \ldots$

$\lambda_{8,1} = 9.0390, \quad \lambda_{8,2} = 13.4828, \quad \lambda_{8,3} = 17.0859,$

$\lambda_{8,4} = 20.5104, \ldots$

the following values of the reduced coefficients ($\bar{A}_{2m,i} = [E_0^{-1}(1-\nu_0^2)q]^{-1}A_{2m,i}$) are obtained

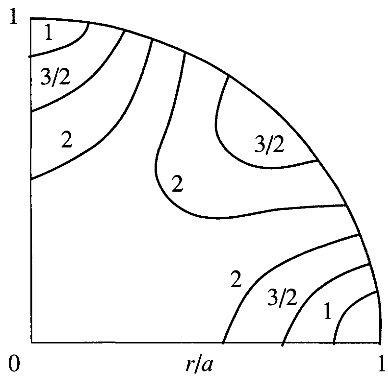

Fig. 1.

$$\bar{A}_{0,1} = 1.431,\ \bar{A}_{0,2} = -0.280,\ \bar{A}_{0,3} = 1.56 \cdot 10^{-2},$$

$$\bar{A}_{0,4} = 2.121 \cdot 10^{-3},\ \bar{A}_{0,5} = 5.13 \cdot 10^{-4}$$

$$\bar{A}_{4,1} = 4.991;\quad \bar{A}_{4,2} = 0.269,$$

$$\bar{A}_{4,3} = 4.13 \cdot 10^{-2};\quad \bar{A}_{4,4} = 1.35 \cdot 10^{-2}$$

$$\bar{A}_{8,1} = 0.483;\quad \bar{A}_{8,2} = 6.02 \cdot 10^{-2};$$

$$\bar{A}_{8.3} = 1.426 \cdot 10^{-2};\quad \bar{A}_{8,4} = 6.28 \cdot 10^{-3} \ldots$$

The lines of equal settlement for the plate being considered are shown in Fig. 1. In view of the symmetry, only a quarter of it is shown. The numbers on the lines correspond to the values of the settlements divided by the quantity $q(1 - \nu_0^2)/E_0$.

Questions concerning the existence and convergence of the solutions of truncated system of the form of (9) have not been considered here since it has been proved that the mechanism employed above to implement the method of orthogonal polynomials (Ref. 3, p. 55,56) leads to regular infinite systems which can be solved by the truncation method.

In the case of another model of the elastic base, the kernel $K(r, \rho, \varphi, \theta)$ has to be represented[2] in the form of the sum of the Boussinesq solution and a certain smooth function that can be expanded in spherical and trigonometric functions, which somewhat changes the form of the equations for determining the coefficients H_{2m}^{4n} and $A_{2m,i}$ in relations (4) and (7). However, the procedure for constructing the solution remains as before.

The proposed procedure for solving the contact problem in the theory of elasticity by Ritz's method using Rektorys' approach enables one to avoid tedious intermediate calculations of the total energy functional of the "plate + elastic base + external load" system and the need to differentiate it with respect to the coefficients of the basis functions when obtaining the system of resolvents.

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